BTECH NOTES SERIES

Fluid Mechanics/Fluid Mechanics and Machines

(As Per AICTE/Technical Universities Syllabus)

CONTENT

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FLUID KINEMATICS & COMPRESSIBLE FLOW

TYPES OF FLOW IN A PIPE

When a fluid is flowing in a pipe, the innumerable small particles get together and form the flowing stream. Through there are many types of flows yet the flowing are important from the subject point of view:

Uniform Flow

A flow in which the velocities of liquid particle at all sections of the pipe or channel are equal, is called a uniform flow, this term is generally applied to flow in channels.

$$\frac{\partial V}{\partial t} = 0$$

Non-Uniform Flow

A flow in which the velocities of liquid particles at all sections of the pipe or channel are not equal, is called a non uniform flow.

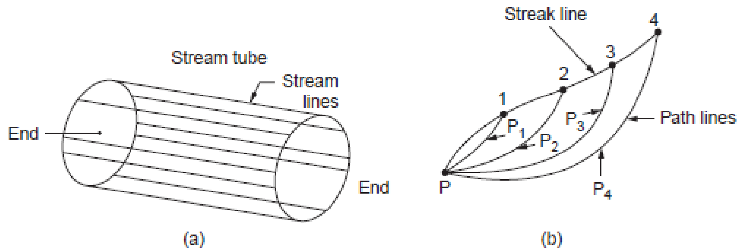
Streamline Flow

A flow in which the particles has a definite path and the paths of individual particles do not cross each other, is called a stream line flow. It is also called a laminar flow.

Path Line and Streak Line

Path line is the trace of the path of a single particle over a period of time. Path line shows the direction of the velocity of a particle at successive instants of time. In steady flow path lines and stream lines will be identical.

Streak lines provide an instantaneous picture of the particles, which have passed through a given point like the injection point of a dye in a flow. In steady flow these lines will also coincide with stream lines.



Particles P_1, P_2, P_3, P_4, starting from point P at successive times pass along path lines shown. At the instant of time considered the positions of the particles are at 1, 2, 3 and 4. A line joining these points is the streak line.

Turbulent Flow

A flow in which each liquid particle does not have a definite path, and the paths of individual particles also cross each other, is called a turbulent flow.

Steady Flow

A flow in which the quantity of liquid flowing per second is constant, is called a steady flow. A steady flow may be uniform or non-uniform.

$$\frac{\partial V}{\partial t} = 0, \frac{\partial p}{\partial t} = 0, \frac{\partial \rho}{\partial t} = 0$$

Unsteady Flow

A flow in which the quantity the of liquid flowing per second is *not* constant i.e. flow parameters changes with time, is called unsteady flow.

$$\frac{\partial V}{\partial t} \neq 0$$

Laminar and Turbulent Flow

If the flow is smooth and if the layers in the flow do not mix macroscopically then the flow is called **laminar flow**. For example a dye injected at a point in laminar flow will travel along a continuous smooth line without generally mixing with the main body of the fluid. Momentum, heat and mass transfer between layers will be at molecular level of pure diffusion. In laminar flow layers will glide over each other without mixing.

In **turbulent flow** fluid layers mix macroscopically and the velocity/temperature/mass concentration at any point is found to vary with reference to a mean value over a time period.

For example $u = \bar{u} + u'$ where u is the velocity at an instant at a location and $\bar{u}$ is the average velocity over a period of time at that location and u′ is the fluctuating component. This causes higher rate of momentum/heat/mass transfer. A dye injected into such a flow will not flow along a smooth line but will mix with the main stream within a short distance.

The difference between the flows can be distinguished by observing the smoke coming out of an incense stick. The smoke in still air will be found to rise along a vertical line without mixing. This is the laminar region. At a distance which will depend on flow conditions the smoke will be found to mix with the air as the flow becomes turbulent. Laminar flow will prevail when viscous forces are larger than inertia forces. Turbulence will begin where inertia forces begin to increase and become higher than viscous forces.

Compressible Flow

A flow in which the volume and thus the density of the flowing fluid changes during the flow, is called a compressible flow. All the gases are generally considered to have compressible flows.

Incompressible Flow

A flow in which the volume and thus the density of the flowing fluid does not change during the flow, is called an incompressible,. All the liquids are, generally, considered to have incompressible flow.

Rotational flow

A flow in which the fluid particles also rotate (i.e. have some angular velocity) about their own axes, while flowing is called a rotational flow. Various rotational components are

$$w_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$w_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$w_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

For *rotational* flows, one or more terms of w_x, w_y, w_z is different from zero.

Irrotational Flow (Potential flow)

A flow in which the fluid particles do not rotate about their do not rotate about their own axes, and retain their original orientations, is called an irrotational flow.

For irrotational flow

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = 0\,;\ \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = 0$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

Example

For the flow, $u = xy^3z$, $v = -y^2z^2$, $w = yz^2-(y^3z^2/2)$, determine the components of rotation about the various axes.

Solution

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}\left(0 - 3xy^2z\right) = -\frac{3}{2}xy^2z$$

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = \frac{1}{2}\left(z^2 - \frac{3y^2z^2}{2} + 2y^2z\right)$$

and $$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = \frac{1}{2}\left(xy^3 - 0\right) = \frac{1}{2}xy^3$$

Example

The velocity components in a two-dimensional velocity field for an incompressible fluid are expressed as

$$u = \frac{y^3}{3} + 2x - x^2y$$

$$v = xy^2 - 2y - \frac{x^3}{3}$$

Show that these functions represent a possible case of an irrotational flow.

Solution

$$\frac{\partial u}{\partial x} = 2 - 2xy$$

and $$\frac{\partial v}{\partial y} = 2xy - 2$$

So that $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2 - 2xy + 2xy - 2 = 0$$

Therefore they represent a possible case of fluid flow. The rotation w of any fluid element in the flow field is,

$$w = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(xy^2 - 2y - \frac{x^3}{3}\right) - \frac{\partial}{\partial y}\left(\frac{y^3}{3} + 2x - x^2 y\right)\right]$$

$$= \frac{1}{2}\left[(y^2 - x^2) - (y^2 - x^2)\right] = 0$$

Hence, the flow is irrotational.

Hence, we can say that these functions represent a possible case of an irrotational flow.

One-Dimensional Flow

A flow whose streamline may be represented by a straight line is called one dimensional flow. It is because of the reason that a straight streamline, being a mathematical line, possesses on dimension only; i.e. either x-x, or y-y or z-z direction. 1D flow examples include flow through a pipe or flow along a river.

Two-Dimensional Flow

A flow, whose streamlines may be represented by a curve, is called a two dimensional flow. It is because of the reason that a curved streamline will be along any two mutually perpendicular directions. 2D flow examples include flow over a flat surface or flow over an air foil.

Three-dimensional Flow

A flow, whose streamlines may be represented in space i.e., along three mutually perpendicular directions, is called three dimensional flow. 3D flow examples include flow in a tornado or flow around a three-dimensional object like a car or airplane.

STREAM LINE

In a fluid flow, a continuous line is so drawn that it is tangential to the velocity vector at every point is known as streamline.

There can be no flow across the stream line, as the velocity perpendicular to the stream line is zero at all points. The flow along the stream line can be considered as one dimensional flow, though the stream line may be curved as there is no component of velocity in the other directions. Stream lines define the flow paths of streams in the flow. The flow entering between two stream lines will always flow between the lines. The lines serve as boundaries for the stream.

If the velocity vector **V** = **i**u + **j**v + **k**w then the differential equation of a streamline is given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

Example

*If for a flow, V = 3x**i** - 3y**j**, then find equation of streamline through (1,1).*

Solution

u = 3x and v = -3y

The equation of a streamline in two dimensional flow is

$$\frac{dx}{u} = \frac{dy}{v}$$

Here $\frac{dx}{3x} = -\frac{dy}{3y}$

On integration

$$\frac{1}{3}\ln x = -\frac{1}{3}\ln y + \frac{1}{3}\ln C$$

where C is constant.

Or $\ln xy = \ln C \Rightarrow xy = C$

For the streamline passing through (1,1), C = 1.

Hence xy = 1

Example (BPUT 2020, 6 marks)

The velocity field in a fluid medium is given by V = axi + ayj + (-2az)k. Find the equation of streamline at point P (2, 2, 4).

Solution

Given data: V = axi + ayj + (-2az) and P(2, 2, 4).

From this u = ax; v = ay; w = (-2az)

Now dx/u = dy/ay = dz/(-2az)

Consider expression of dx and dy only, we have

$$\frac{dx}{ax} = \frac{dy}{ay} \Rightarrow \int \frac{dx}{ax} = \int \frac{dy}{ay} \Rightarrow \log_e x = \log_e y + \log_e c \Rightarrow x = cy$$

Hence at point P(2, 2, 4), c = 1 ⇒ x = y

Similarly consider expression of dx and dz only, we have

$$\frac{dx}{ax} = \int \frac{dz}{-2az} \Rightarrow \log_e x = \frac{-1}{2}\log_e z + \log_e c \Rightarrow x = c/\sqrt{z}$$

∴ At point P (2, 2, 4) c = 4, x = 4/√z

Therefore, we get the stream line equation as x = y = 4/√z

Example (GTU 2022, 3 marks)

Obtain the equation to the streamlines for the velocity field given as: $V = 2x^3i - 6x^2yj$.

Solution

Given that $V = 2x^3i - 6x^2yj$

Here $u = 2x^3$; $v = 6x^2y$

The streamlines in two dimensions are defined by

dx/u = dy/v

Or $$\frac{dy}{dx} = \frac{v}{u} = \frac{-6x^2y}{2x^3} = \frac{-3y}{x}$$

Separating the variables

dy/y = -3dx/x

Integrating $\ln(y) = -3\ln(x) + C_1$

Or $yx^3 = C$

ACCELERATION

Acceleration is a vector.

If $V = iu + jv + kw$

then acceleration a_x, a_y and a_z in x, y and z directions are

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

Example

The velocity vector in an incompressible flow is given by

$V = (6xt + yz^2)\mathbf{i} + (3t + xy^2)\mathbf{j} + (xy - 2xyz - 6tz)\mathbf{k}$

(i) Verify whether the continuity equation is satisfied.

(ii) Determine the acceleration vector at point (1,1,1) at t = 1.

Solution

(i) Given $V = (6xt + yz^2)\mathbf{i} + (3t + xy^2)\mathbf{j} + (xy - 2xyz - 6tz)\mathbf{k} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$

We see $u = 6xt + yz^2$

$\therefore$ $\frac{\partial u}{\partial x} = 6t$

Similarly $v = 3t + xy^2$

$$\frac{\partial v}{\partial y} = 2xy$$

$$w = xy - 2xyz - 6tz$$

$$\frac{\partial w}{\partial z} = -2xy - 6t$$

Now $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 6t + 2xy - 2xy - 6t = 0$

Hence the continuity equation is satisfied.

(ii) Acceleration

$$a = a_x i + a_y j + a_z k$$

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$= 6x + (6xt + yz^2)(6t) + (3t + xy^2)(z^2) + (xy - 2xyz - 6tz)(2yz)$$

At A (1,1,1) and t = 1

$$a_x = 6 + (6+1)(6) + (3+1)(1) + (1-2-6)(2) = 38 \text{ units}$$

Also $a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$

$$= 3 + (6xt + yz^2)(y^2) + (3t + xy^2)(2xy) + (xy - 2xyz - 6tz)(0)$$

At point A (1,1,1) and at t = 1

$$a_y = 3 + (6+1)(1) + (3+1)(2) = 18 \text{ units}$$

Similarly $a_z = 39$ units

Hence at A (1,1,1) and at t = 1

$$\mathbf{a} = 38\mathbf{i} + 18\mathbf{j} + 39\mathbf{k}$$

Example

Velocity for a two dimensional flow field is given by V = (3 + 2xy + 4t²)I + (xy² + 3t)j. Find the velocity and acceleration at a point (1, 2) after 2 seconds.

Solution

Velocity

Velocity for a two dimensional field

$$V = (3 + 2xy + 4t^2)i + (xy^2 + 3t)j$$

$$V = u\bar{i} + v\bar{j}$$

Now $$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(3 + 2xy + 4t^2) = 2y$$

$$\frac{\partial u}{\partial x} \, at\, (1,2) = 2(2) = 4$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(\mathrm{xy}^2 + 3\mathrm{t}) = 2xy$$

$$\frac{\partial u}{\partial y} \, at\, (1,2) = 2(1)(2) = 4$$

$\therefore$ $$V = 4\bar{i} + 4\bar{j}$$

Magnitude of velocity

$$V = \sqrt{4^2 + 4^2} = 5.65\, m/s$$

Acceleration

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial x} = 2y; \frac{\partial u}{\partial y} = 2x; \frac{\partial u}{\partial t} = 8t$$

$$a_x = (3 + 2xy + 4t^2)2y + (xy^2 + 3t)(2x) + 8t$$

$$= 6y + 4xy^2 + 8yt^2 + 2x^2y^2 + 6xt + 8t$$

$$a_{x(1,2)} = 6(2) + 4(1)(2)^2 + 8(2)t^2 + (2)(1)^2(2)^2 + 6(1)(t) + 8t$$

$$= 36 + 14t + 16t^2$$

At t = 2 sec. $a_{x(t=2)} = 36 + 28 + 64 = 128$ m/s^2

Example

A velocity field is given by u = - 3x, v = - 2y, w = z. Is this flow steady? Is it two- or three-dimensional? At (x, y, z) = (1, 1, 1), compute (a) the velocity, (b) the local acceleration, and (c) the convective acceleration.

Solution

The velocity field is given as

$$V = -3xi + 2yj + zk$$

The flow is steady since V is independent of time t. The flow is three-dimensional since all the three components of velocity are finite.

Velocity

At (1, 1, 1) V = -3i +2j + k

Local acceleration

$$\frac{\partial V}{\partial t} = 0$$

Convective acceleration

$$\frac{du}{dt}i + \frac{dv}{dt}j + \frac{dw}{dt}k$$

Now $$\frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -3x(-3) + 2y(0) + z(0) = 9x$$

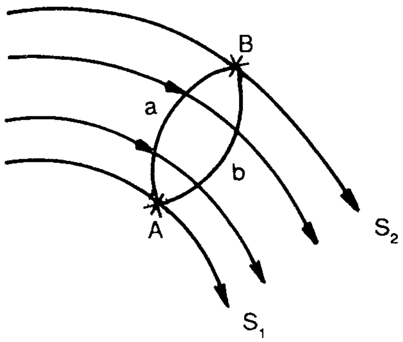
$$\frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -3x(0) + 2y(2) + z(0) = 4y$$

and $$\frac{dw}{dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -3x(0) + 2y(0) + z(1) = z$$

Thus, the convective acceleration at (1, 1, 1) is 9(1)i + 4(1)j + 1k = 9i + 4j + k

STREAM FUNCTION

In a two dimensional flow consider two streamlines S_1 and S_2. A stream function ψ is so defined that it is constant along a streamline and the difference of ψ for two streamlines is equal to the flow rate between them.



Thus $\psi_B - \psi_A =$ flow rate between S_1 and S_2. The flow left to right is taken as positive, in the sign convention. The velocities u and v in x and y directions are given by

$$u = \frac{\partial \psi}{\partial y}$$

and $$v = -\frac{\partial \psi}{\partial x}$$

The stream function ψ is defined as above for two dimensional flows only.

For an irrotational flow

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

$\therefore$ $$-\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = 0$$

It means that Laplace equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

is satisfied by the stream function in *irrotational* flow.

Conversely, if ψ does not satisfy $\nabla^2 \psi = 0$, then the flow is *rotational*.

Example

A stream function is given by the expression $\psi = 2x^2 - y^3$*. Find the components of velocity, as well as the resultant velocity at a point P (3,1).*

Solution

Stream function in the problem is given as

$$\psi = 2x^2 - y^3$$

Co-ordinates of point P, x = 3 and y = 1

Using the relation

$$u = \frac{\partial\psi}{\partial y}$$

We get $$u = \frac{\partial}{\partial y}\left(2x^2 - y^3\right) = -3x1^2 = -3$$

Now using the relation

$$v = -\frac{\partial\psi}{\partial x}$$

We get $$v = -\frac{\partial}{\partial x}\left(2x^2 - y^3\right) = -4x = -4x3 = -12$$

Example (AKTU 2023, 10 marks)

Describe the method of determination of the stream function given the velocity relationship and also determine the stream function given $u = 4xy$ *and* $v = c - 2y^2$.

Solution

Check for continuity

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0$$

$$u = 4xy;\ v = c - 2y^2$$

$$\frac{\partial u}{\partial x} = 4y; \frac{\partial v}{\partial y} = 4xy$$

∴ We see that continuity is satisfied.

Let $u = f_1(x, y)$

$\therefore \qquad u = \frac{\partial \psi}{\partial y} = f_1(x, y)$

$\therefore \qquad \psi = \int f_1(x, y)dy + f(x)$

where the second terms is a function of x only.

Let $\qquad -v = \frac{\partial \psi}{\partial x} = f_2(x, y)$

$$\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x}\left[\int f_1(x, y)dy\right] + f'(x) = f_2(x, y) = -v$$

Comparing the terms with f_2 (x, y), f ′ (x) can be obtained

$$\psi = \int f(x)dx + \int f_1(x, y)dy + \text{constant}$$

$$u = \frac{\partial \psi}{\partial y} = 4xy$$

$\therefore \qquad \psi = \int 4xydy = 2xy^2 + f(x) \qquad (1)$

where f(x) is a function of x only.

$$\frac{\partial \psi}{\partial x} = -v = 2y^2 - c = 2y^2 + f'(x) \text{ using (1)}$$

Differentiating (1) wrt x and comparing f'(x) = -c, f(x) = -cx

Now substitute for f(x) in (1)

$\psi = 2xy^2 - cx + \text{constant} \qquad (2)$

Check (use equation 2)

$$u = \frac{\partial \psi}{\partial y} = 4xy; v = -\frac{\partial \psi}{\partial x} = -2y^2 + c$$

Problem

For the following stream functions, calculate the velocity at a point (2,3)

(a) $\psi = 2xy$

(b) $\psi = 3x^2y - y^3$

Answer: (a) √52 units (b) 39 units

Example (JNTUH 2021, 15 marks)

A uniform flow of 10 m/s is flowing over a doublet of strength 15 m^2/s. The doublet is in the line of the uniform flow. The polar co-ordinates of a point P in the flow field are 0.9 m and 30^0. Find: (a) Stream line function and (b) the resultant velocity at the point.

Solution

Given data

U = 10 m/s; μ = 15 m^2/s; r = 0.9 m; $\theta = 30^0$

Radius R of Rankine circle

$$R = \sqrt{\frac{\mu}{2\pi U}} = \sqrt{\frac{15}{2\pi x10}} = 0.488\,m$$

The polar co-ordinates of the point P are 0.9 m and 30°.

Hence r = 0.9 m and 0 = 30°.

As the value of r is more than the radius of the Rankine circle, hence point P lies outside the cylinder.

Value of stream line function at the point P

The stream line function for the composite flow at any point is given by equation

$$\psi = U\left(r - \frac{R^2}{r}\right)\sin\theta = 10\left(0.9 - \frac{0.488^2}{0.9}\right)\sin 30^0 = 3.177\,m^2/s$$

Resultant velocity at the point P

The radial velocity and tangential velocity at any point in the flow field are given by equations

$$u_r = U\left(1 - \frac{R^2}{r^2}\right)\cos\theta = 10\left(1 - \frac{0.488^2}{0.9^2}\right)\cos 30^0 = 611\,m/s \text{ (outward)}$$

positive sign shows that radial velocity is outward.

and $$u_\theta = -U\left(1 + \frac{R^2}{r^2}\right)\sin\theta = -10\left(1 + \frac{0.488^2}{0.9^2}\right)\sin 30^0 = -6.47\,m/s$$

minus sign shows that tangential velocity is clockwise.

Resultant velocity

$$V = \sqrt{u_r^2 + u_\theta^2} = \sqrt{6.11^2 + (-6.47)^2} = 8.89\,m/s$$

VELOCITY POTENTIAL

In irrotational flows, the velocity can be written as a negative gradient of a scalar function ϕ called velocity potential.

$$u = -\frac{\partial \phi}{\partial x}$$

$$v = -\frac{\partial \phi}{\partial y}$$

and $$w = -\frac{\partial \phi}{\partial z}$$

For an incompressible flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

which after putting values of u, v and w becomes

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

Thus the velocity potential satisfies the Laplace equation. Conversely, any function ϕ which satisfies the Laplace equation is a possible irrotational flow case.

We also see that

$$u = -\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

and $$v = -\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

Example (AKTU 2023, 10 marks)

Prove that the stream function and potential function lead to orthogonality of stream lines and equipotential flow lines.

Solution

$$\psi = \psi(x, y)$$

$\therefore$ $$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

Substituting from definition of $\partial \psi / \partial x$ and $\partial \psi / \partial y$ as -v and u.

$$\partial\psi = -vdx + udy$$

as ψ is constant along a stream line dψ = 0,

∴ vdx = udy

The slope of the stream line at this point is thus given by

$$\frac{dy}{dx} = \frac{v}{u} \quad (10$$

Similarly ϕ = ϕ (x, y)

∴ $$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy$$

Substituting for $\partial\phi/\partial x$ and $\partial\phi/\partial y$ as -u and -v

dϕ = -udx - vdy

as dϕ = 0 along an equipotential line. udx = -vdy

∴ $$\frac{\partial y}{\partial x} = -\frac{u}{v} \quad (2)$$

These values of slopes show that the two sets of lines are perpendicular to each other. Hence stream lines and equipotential lines are orthogonal.

Example

Verify whether the following functions are valid potential functions:

(i) ϕ = A xy *(ii) ϕ = m 1n x*

Solution

A valid potential function satisfies the Laplace equation.

(i) ϕ = A xy

$$\frac{\partial\phi}{\partial x} = A \text{ and } \frac{\partial\phi}{\partial y} = Ax$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 + 0 = 0$$

Hence ϕ = Axy is a valid potential function

(ii) ϕ = m ln x

$$\frac{\partial\phi}{\partial x} = \frac{m}{x} \text{ and } \frac{\partial\phi}{\partial y} = 0$$

$$\frac{\partial^2\phi}{\partial x^2} = -\frac{m}{x^2} \text{ and } \frac{\partial^2\phi}{\partial y^2} = 0$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial x^2} = -m/x^2 \neq 0$$

Problem

Verify whether the following functions are valid potential functions:

(i) $\phi = A(x^2 - y^2)$ *(ii)* $\phi = A \cos x$

Answer: (i) valid (ii) not valid

Example

Values of ϕ *for various flows are given. determine the corresponding values of* ψ *.*

(i) $\phi = 3xy$ *(ii)* $\phi = 4(x^2 - y^2)$

Solution

(i) $\phi = 3xy$

$$u = -\frac{\partial\phi}{\partial x} = -3y = \frac{\partial\psi}{\partial y}$$

Hence $\psi = -\frac{3}{2}y^2 + f(x)$

$$v = -\frac{\partial\phi}{\partial y} = -3x = -\frac{\partial\psi}{\partial x} = -f'(x)$$

$f'(x) = 3x$ and hence $f(x) = \frac{3}{2}x^2 + c$

$\therefore$ $\psi = \frac{3}{2}\left(x^2 - y^2\right) + c$ where c = constant

(ii) $\phi = 4(x^2 - y^2)$

$$u = -\frac{\partial\phi}{\partial x} = -8x = \frac{\partial\psi}{\partial y}$$

Hence $\psi = -8xy + f(x)$

$$v = -\frac{\partial\phi}{\partial y} = +8y = -\frac{\partial\psi}{\partial x} = +8y - f'(x)$$

Hence f'(x) = 0 and f(x) constant = c

$\therefore \quad \psi = -8xy + c$

Problem

Determine the corresponding values of ψ *, if* $\phi = x + y + 3$*.*

Answer: $\psi = x - y + c$

Example

Explain how the potential function can be obtained if the stream function for the flow is specified.

Solution

(1) Irrotational nature of the flow should be checked first. Stream function may exist, but if the flow is rotational potential function will not be valid.

(2) The values of *u* and *v* are obtained from the stream function as

$$\frac{\partial \psi}{\partial y} = u$$

$$\frac{\partial \psi}{\partial x} = -v$$

(3) From the knowledge of *u* and *v*, ϕ can be determined using the same procedure as per the determination of stream function

$$u = -\frac{\partial \phi}{\partial x}$$

$$\phi = -\int u dx - f(y)$$

where *f*(*y*) is a function of *y* only

$\partial \phi / \partial y$ is determined and equated to – v

Comparing $f'(y)$ is found and then *f*(*y*) is determined and substituted in equation *A*

$$\phi = -\int u dx - f(y) + C$$

Example

For the following stream functions, determine the potential function

(i) $\psi = (3/2)(x^2 - y^2)$

(ii) $\psi = -8xy$

(iii) $\psi = x - y$

Solution

(i) Step1: We have

$$u = \frac{\partial \psi}{\partial y} = -3y$$

$$-v = \frac{\partial \psi}{\partial x} = 3x \Rightarrow v = -3x$$

To check for irrotational flow

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

here, both are -3, so checks

Step 2: $u = -3y$, also $u = -\frac{\partial \phi}{\partial x}$

$\therefore \quad \phi = \int 3y dx + f(y)$

$$\phi = 3xy + f(y) \qquad \text{(A)}$$

Diff. eq. (A) w.r.t y and equating to v,

$$\frac{\partial \phi}{\partial y} 3x + f' y = -v = 3x$$

$\therefore \quad f'(y) = 0$ and so f(y) = constant.

Substituting in A,

$$\phi = 3xy + cons\tan t$$

Step3: Check $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \frac{\partial \phi}{\partial y} = 3y, \frac{\partial^2 \phi}{\partial x^2} = 0$

So also $\frac{\partial^2 \phi}{\partial y^2} = 0$

So checks

$$u = -\frac{\partial \phi}{\partial x} = -3y, v = -\frac{\partial \phi}{\partial y} = -3x \text{ also checks.}$$

(ii) Do yourself. $\phi = 4x^2 - 4y^2$

(iii) Do yourself. $\phi = x + y$

Example (BPUT 2020, 16 marks)

In a two dimensional incompressible flow, the fluid velocity components are given by u = x - 4y, v = -y - 4x. Show that the velocity potential exists and determine its form for stream function as well.

Solution

Velocity Potential

Given u = x - 4y and v = -y - 4x

∴ $\partial u/\partial x = 1$ and $\partial v/\partial y = -1$

∴ $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 1 - 1 = 0$$

Hence flow is continuous and velocity potential exists.

let ϕ = velocity potential

let velocity components in terms of velocity potential is given by

$$\frac{\partial \phi}{\partial x} = -u = -(x - 4y) = -x + 4y \quad (1)$$

and $$\frac{\partial \phi}{\partial y} = -v = -(-y - 4x) = y + 4x \quad (2)$$

Integrating (1)

$$\phi = -\frac{x^2}{2} + 4xy + C \quad (3)$$

where C is a constant of integration, which is independent of x.

This constant can be a function of y.

Differentiating the above equation with respect to "y", we get

$$\frac{\partial \phi}{\partial y} = 0 + 4x + \frac{\partial C}{\partial y}$$

But from equation (3), we have

$$\frac{\partial \phi}{\partial y} = y + 4x$$

Equating the two values of $\partial\phi/\partial y$, we get

$$4x + \frac{\partial C}{\partial y} = y + 4x \Rightarrow \frac{\partial C}{\partial y} = y$$

Integrating the above equation, we get

$$C = \frac{y^2}{2} + C_1$$

where C_1 is a constant of integration, which is indenpendent of x and y.

Taking it equal to zero, we get C = $y^2/2$.

Substituting this value of C in (3), we get

$$\phi = -\frac{x^2}{2} + 4xy + \frac{y^2}{2}$$

Stream function

let ψ = stream function

The velocity components of stream function are

$$\frac{\partial \psi}{\partial x} = v = -y - 4x \qquad (4)$$

and $$\frac{\partial \psi}{\partial y} = -u = -(x - 4y) = -x + 4y \qquad (5)$$

Integrating equation (4) wrt x, we get

$$\psi = -yx - \frac{4x^2}{2} + K \qquad (6)$$

where K is a constant of integration which is independent of x but can be a function of y.

Differentiating equation (6) wrt y, we get

$$\frac{\partial \psi}{\partial y} = -x - 0 + \frac{\partial K}{\partial y}$$

But from eq. (5)

$$\frac{\partial \psi}{\partial y} = -x + 4y$$

Equating the two values of $\partial\psi/\partial y$, we get

$$-x + \frac{\partial K}{\partial y} = -x + 4y$$

Or $$\frac{\partial K}{\partial y} = 4y$$

Integrating, we get

$$K = \frac{4y^2}{2} = 2y^2$$

Substituting the value of k in equation (6), we get

$$\psi = -yx - 2x^2 + 2y^2$$

Example (BPUT 2020, 6 marks)

The velocity potential function is given by $5(x^2 - y^2)$. Calculate the velocity components at the point (5, 6).

Solution

$$\phi = 5(x^2 - y^2)$$

$\therefore$ $\partial\phi/\partial x = 10x$

and $\partial\phi/\partial y = -10y$

Velocity components u and v are given as

$$u = -\partial\phi/\partial x = -10x$$

and $v = -\partial\phi/\partial y = -(-10y) = 10y$

The velocity components at (5, 6) i.e. at x = 5 and y = 6 are

$$u = -10(5) = -50 \text{ units}$$

and $v = 10y = 10(6) = 60$ units.

Example (HPTU 2021, 6 marks)

If the expression for the stream function is described by $\psi = x^3 - 3xy^2$, indicate whether the flow is rotational or irrotational. If the flow is irrotational, determine the value of velocity potential.
(a) $\phi = y^3 - 3xy^2$ (b) $\phi = -3x^2y$

Solution

Given that $\psi = x^3 - 3xy^2$ (stream function)

Vorticity vector and check for irrotational flow

A two dimensional flow in x-y plane will be irrotational if the vorticity vector in the z direction is zero.

$\therefore \qquad \Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad (1)$

We know $\quad u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}(x^3 - 3xy^2) = -6xy$

and $\quad v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}(x^3 - 3xy^2) = -3(x^2 - y^2)$

$\therefore \quad \frac{\partial u}{\partial y} = -6; \frac{\partial v}{\partial x} = -6x$

Substituting these values in eq (1), we get

$$\Omega_z = -6x - (-6y) = 0$$

Hence, the flow is irrotational.

Velocity potential

For an irrotational flow Laplace equation in ϕ must be satisfied.

i.e. $\quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$

Let us check the validity for each expression.

(a) $\quad \phi = y^3 - 3xy^2$

$$\frac{\partial^2 \phi}{\partial x^2} = -6y; \frac{\partial^2 \phi}{\partial y^2} = 6y$$

$\therefore \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -6y + 6y = 0$

(b) $\quad \phi = -3x^2 y$

$$\frac{\partial^2 \phi}{\partial x^2} = -6y; \frac{\partial^2 \phi}{\partial y^2} = 0$$

$\therefore \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y} \neq 0$

Hence correct value is $\phi = -3x^2 y$.

RATE OF DISCHARGE

The quantity of a liquid flowing per second through a section of a pipe or a channel, is known as the rate of discharge or simply discharge. It is generally denoted by Q.

Let A = cross sectional area of the pipe and V = average velocity of the fluid

Then discharge

$$Q = \text{Area x Average Velocity} = A.V$$

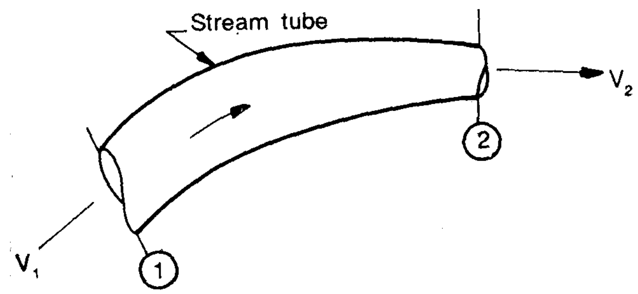
EQUATION OF CONTINUITY OF A LIQUID FLOW

In One Dimensional method

In steady flow, mass rate of flow into a stream tube is equal to mass rate of flow out of the tube

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

For incompressible fluid, under steady flow (see figure)



$$A_1 V_1 = A_2 V_2$$

Proof

Let V_1 = average velocity at section 1

ρ_1 = density at section 1

A_1 = area of duct at section1

Similarly at section 2.

Rate of flow at section 1 = $\rho_1 A_1 V_1$

at section 2 = $\rho_2 A_2 V_2$

According to the law of conservation of mass

Rate of flow at section 1 = rate of flow at section 2

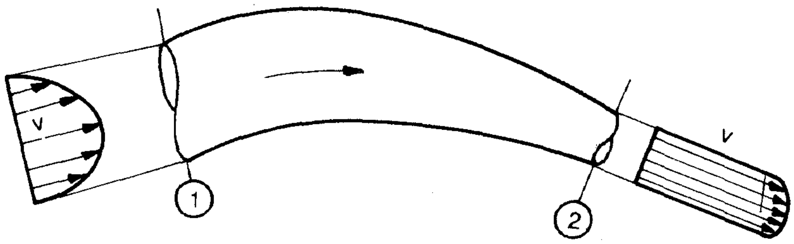
or $\rho_1 A_1 V_1 = \rho_2 A_2 V_2$

This is continuity equation. If $\rho_1 = \rho_2$ (i.e. fluid is incompressible) then

$$A_1 V_1 = A_2 V_2$$

In Differential Form

See figure

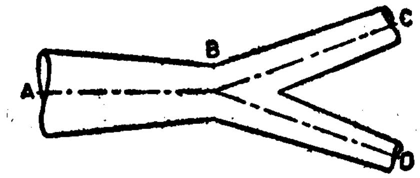


For incompressible flow

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial w}{\partial z} = 0$$

Problem

A pipe AB branches into two pipes C and D as shown in figure above. The pipe has diameter of 45 cm at A, 30 cm at B, 20 cm at C and 15 cm at D. Determine the discharge at A if the velocity at a is 2 m/sec. Also determine the velocities at B and D, if the velocity at C is 4 m/sec.



Solution

Discharge at A

$$Q_A = A_A V_A = \left(\frac{\pi}{4}\right) \text{x } (0.45)^2 \text{ x } 2 = 0.318 \text{ m}^3/\text{sec.}$$

Velocity at B

Since the discharge is continuous , therefore,

$$A_A V_A = A_B V_B$$

$$\Rightarrow \quad V_B = \frac{A_v V_A}{A_B} = \frac{Q}{A_B} = \frac{0.318}{\left(\pi/4\right) \text{x} (0.3)^2} = 4.5 \text{ m/sec.}$$

Velocity at D

From the geometry of flow, we find that the discharge at A

$Q_A = Q_C + Q_D$

$\Rightarrow$ $0.318 = A_cV_c + A_DV_D = \left(\frac{\pi}{4}\right) x (0.2)^2 x\, 4 + \left(\frac{\pi}{4}\right) x (0.15)^2 x\, V_D$

$\Rightarrow$ $V_D = 10.87$ m/sec.

Example

Derive an expression for continuity equation for a three dimensional flow.

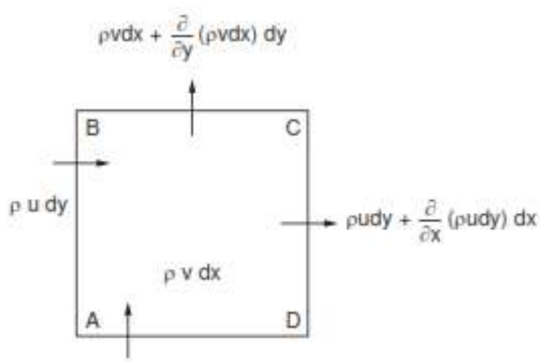
Solution

Consider an element of size dx, dy, dz in the flow as shown in figure.

Applying the law of conservation of mass, for a given time interval,

The net mass flow into the element through all the surfaces

= The change in mass in the element.



First considering the y – z face, perpendicular to the x direction and located at x, the flow through face during time dt is given by

$$\rho.u.dy.dz.dt \qquad (1)$$

The flow through the y – z face at x + dx is given by

$$\rho.u.dy.dz.dt + \frac{\partial}{\partial x}(\rho.u.dy.dz.dt)dx \qquad (2)$$

The net mass flow in the x direction is the difference between the quantities given by (1) and (2) and is equal to

$$\frac{\partial}{\partial x}(\rho.u)\,dx.dy.dz.dt \qquad (3)$$

Similarly the net mass through the faces z – x and x – y in y and z directions respectively are given by

$$\frac{\partial}{\partial x}(\rho.\mathrm{v})\,\mathrm{dx}.dy.dz.dt \qquad (4)$$

$$\frac{\partial}{\partial x}(\rho.\mathrm{w})\,\mathrm{dx}.dy.dz.dt \qquad (5)$$

The change in the mass in the control volume equals the rate of change of density × volume × time or

$$\frac{\partial \rho}{\partial t}\,\mathrm{dx}.dy.dz.dt \qquad (6)$$

The sum of these quantities should equal zero, cancelling common terms dx dy dz dt

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=\frac{\partial \rho}{\partial t} \qquad (7)$$

This is the general equation. For steady flow this reduces to

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0 \qquad (8)$$

For incompressible flow this becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad (9)$$

Example

Calculate the unknown velocity component in $u = A(x^2 + y^2); v = ?$

Solution

$$u = A(x^2 + y^2)$$

$$\therefore \quad \frac{\partial u}{\partial x} = 2Ax = -\frac{\partial v}{\partial y}$$

$$\therefore \quad v = \int -2Axdy = -2Axy + f(x)$$

The exact nature of f(x) will be known if the boundary conditions are known.

Example

Determine the missing component of velocity distribution such that they satisfy continuity equation

$u = ?$

$v = ax^3 - by^2 + cz^2$

$w = bx^3 - cy^2 + az^2x$

Solution

Since $$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$\frac{\partial u}{\partial x}=-\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

$$= -(-2by+2azx)$$

$$= 2by-2azx$$

$$u=\int\frac{\partial u}{\partial x}dx=\int(2by-2azx)dx$$

$$= 2byx-2az\frac{x^2}{2}+f(y,z)$$

$$= 2byx-azx^2+f(y,z)$$

The exact nature of f(y, z) will be known if the boundary conditions are known.

Problem

Calculate the unknown velocity component in the following, so that the equation of continuity is satisfied.

(i) $u = Ae^x; v = ?$

(ii) $u = ?; v = Axy$

Answer: (i) $v=Ae^x y+f(x)$ (ii) $u=-\frac{Ax^2}{2}+f(y)$

COMPRESSIBLE FLOW

All real fluids are compressible to some extent and therefore their density will change with change in pressure or temperature. If the relative change in density Δρ/ρ is small, the fluid can be treated as incompressible, such as air.

CONTINUITY EQUATION

In case of one-dimensional flow, mass per second = ρAV

(where ρ = mass density, A = area of cross-section, V = velocity)

Since the mass or mass per second is constant according to law of conservation of mass, therefore,

$$\rho AV = \text{constant} \qquad (1)$$

Differentiating the above equation, we get

$$d(\rho AV) = 0 \text{ or } \rho d(AV) + AVd\rho = 0$$

or $$\rho(AdV + VdA) + AVd\rho = 0 \text{ or } \rho AdV + \rho VdA + AVd\rho = 0$$

Dividing both sides by ρAV and rearranging, we get

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0 \qquad (2)$$

Eqn. (2) is also known as **equation of continuity in differential form**.

BERNOULLI'S OR ENERGY EQUATION

From Euler equation

$$\frac{dp}{\rho} + VdV + gdz = 0$$

Integrating $$\int \frac{dp}{\rho} + \int VdV + \int gdz = const \qquad (4)$$

or $$\int \frac{dp}{\rho} + \frac{V^2}{2} + gz = const \qquad (5)$$

In compressible flow since ρ is not constant it cannot be taken outside the integration sign.

In compressible fluids the pressure (p) changes with change of density (ρ), depending on the type of process. Let us find out the Bernoulli's equation for isothermal and adiabatic processes.

Bernoulli's or Energy equation for isothermal process

In case of isothermal process,

pv = constant or p/ρ = constant = c_1 (say) here v is specific volume = 1/ρ

$$\therefore \qquad \rho = \frac{p}{c}$$

$$\therefore \qquad \int \frac{dp}{p} = \int \frac{dp}{p/c_1} = \int \frac{c_1 dp}{p} = c_1 \int \frac{dp}{p} = c_1 \log_e p = \frac{p}{\rho} \log_e p$$

Substituting the value of $\int \frac{dp}{p}$ in eq (5)

$$\frac{p}{\rho} \log_e p + \frac{V^2}{2} + gz = const$$

Dividing both sides by g, we get

$$\frac{p}{\rho g}\log_e p + \frac{V^2}{2g} + z = const \qquad (6)$$

This is Bernoulli's equation for compressible flow undergoing **isothermal process**.

Bernoulli's equation for adiabatic process

In case of an adiabatic process,

$$pv^\gamma = const \Rightarrow \frac{p}{\rho^\gamma} = const = c_2$$

$$\therefore \qquad \rho^\gamma = \frac{p}{c_2} \Rightarrow \rho = \left(\frac{p}{c_2}\right)^{1/\gamma}$$

$$\therefore \qquad \int \frac{dp}{\rho} = \int \frac{dp}{(p/c_2)^{1/\gamma}} = (c_2)^{1/\gamma}\int \frac{1}{p^{1/\gamma}}dp = (c_2)^{1/\gamma}\int p^{-1/\gamma}dp$$

$$= (c_2)^{1/\gamma}\left[\frac{p^{-\frac{1}{\gamma}+1}}{\left(-\frac{1}{\gamma}+1\right)}\right] = \frac{\gamma}{\gamma-1}(c_2)^{1/\gamma}(p)^{\left(\frac{\gamma-1}{\gamma}\right)}$$

Solving, we get

$$\int \frac{dp}{\rho} = \left(\frac{\gamma}{\gamma-1}\right)\frac{p}{\rho}$$

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p}{\rho} + \frac{V^2}{2} + gz = constant$$

Dividing both sides by g, we get

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p}{\rho g} + \frac{V^2}{2g} + z = constant \qquad (7)$$

Eqn. (7) is the Bernoulli’s equation for compressible flow undergoing **adiabatic process**.

Example

A gas with a velocity of 300 m/s is flowing through a horizontal pipe at a section where pressure is 78 kN/m^2 absolute and temperature 40°C. The pipe changes in diameter and at this section, the pressure is 117 kN/m^2 absolute. Find the velocity of the gas at this section if the flow of the gas is adiabatic. Take R = 287 J/kg K and γ = 1.4.

Solution

Given data

Section 1 :

Velocity of the gas, V = 300 m/s

Pressure, p1 = 78 kN/m^2

Temperature, T_1 = 40 + 273 = 313 K

Section 2 :

Pressure, p_2 = 117 kN/m^2

R = 287 J/kg K, γ = 1.4

Velocity of gas at section 2

Applying Bernoulli's equations at sections 1 and 2 for adiabatic process, we have

Applying Bernoulli's equations at sections 1 and 2 for adiabatic process, we have

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1 g}+\frac{V_1^2}{2g}+z_1=\left(\frac{\gamma}{\gamma-1}\right)\frac{p_2}{\rho_2 g}+\frac{V_2^{\ 2}}{2g}+z_2 \qquad \text{(from eq 7)}$$

But $z_1 = z_2$ because pipe is horizontal.

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1 g}+\frac{V_1^2}{2g}=\left(\frac{\gamma}{\gamma-1}\right)\frac{p_2}{\rho_2 g}+\frac{V_2^{\ 2}}{2g}$$

Solving
$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left(1-\frac{p_2}{p_1}x\frac{\rho_1}{\rho_2}\right)=\frac{V_2^{\ 2}}{2}-\frac{V_1^2}{2} \qquad (1)$$

For an adiabatic flow

$$\frac{p_1}{\rho_1}=\frac{p_2}{\rho_2^{\ \gamma}}\Rightarrow\frac{p_1}{p_2}=\left(\frac{\rho_1}{\rho_2}\right)^{\gamma}\Rightarrow\frac{\rho_1}{\rho_2}=\left(\frac{p_1}{p_2}\right)^{1/\gamma}$$

Substituting the value of ρ_1/ρ_2 in eqn (i), we get

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left[1-\frac{p_2}{p_1}\left(\frac{p_1}{p_2}\right)^{1/\gamma}\right]=\frac{V_2^{\ 2}}{2}-\frac{V_1^2}{2}$$

Solving
$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right]=\frac{V_2^{\ 2}-V_1^2}{2} \qquad (ii)$$

Section 1

$$\frac{p_1}{\rho_1} = RT_1 = 287x313 = 89831$$

$$\frac{p_2}{p_1} = \frac{117}{78} = 1.5$$

V_1 = 300 m/s

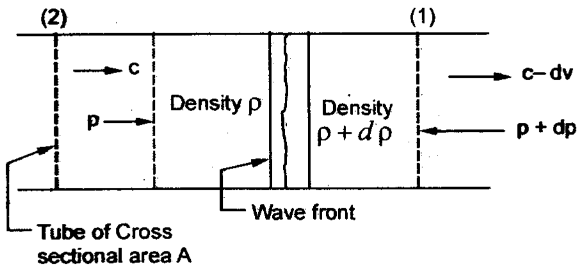
Substituting the values in eq (i), we get

$$\left(\frac{1.4}{1.4-1}\right)x89831\left[1-(1.5)^{\frac{1.4-1}{1.4}}\right] = \frac{V_2^{\,2}}{2} - \frac{300^2}{2}$$

Solving V_2 = 113.05 m/s

SOUND WAVE IN COMPRESSIBLE FLOW

With reference to following figure in which a sound wave os flowing with velocity c.



Applying continuity equation between (1) and (2)

$$\rho cA = (\rho + d\rho)(c - dV)A$$

Neglecting product of small quantities,

$$\rho c = \rho c - \rho dV + cd\rho$$

or $\rho dV = cd\rho$ (1)

Applying momentum equation

Force in right direction = mass x change in velocity from (1) and (2)

or $\rho A - (p + dp)A = \rho cA\{(c - dv) - c\}$

or $-d\rho = -\rho cdV$

or $\rho dV = \dfrac{dp}{c}$

Substituting for ρdV in equation (1)

$$\frac{dp}{c} = cd\rho$$

or $$c^2 = \frac{dp}{d\rho}$$

or $$c = \sqrt{\frac{dp}{d\rho}}$$

This is required expression.

If we want an expression in terms of bulk modulus K then put dp/dρ = K/ρ in above equation.

We will get $$c = \sqrt{\frac{dp}{d\rho}} = \frac{K}{\rho}$$

For isothermal process

p/ρ = constant

or $$\ln p - \ln \rho = \ln(const)$$

or $$\frac{dp}{p} - \frac{dp}{p} = 0 \text{ taking diff}$$

or $$\frac{dp}{d\rho} = \frac{p}{\rho} = RT$$

$\therefore$ from $c = \sqrt{\frac{dp}{d\rho}}$

$$c = \sqrt{\frac{dp}{d\rho}} = \sqrt{\frac{p}{\rho}} = \sqrt{RT}$$

For adiabatic process

$$\frac{p}{\rho^\gamma} = const$$

or $$\ln \rho - \gamma \ln \rho = \ln const$$

or $$\frac{dp}{p} - \gamma \frac{d\rho}{\rho} = 0$$

or $$\frac{dp}{d\rho} = \gamma \frac{p}{\rho} = \gamma RT$$

$\therefore$ $$c = \sqrt{\gamma RT}$$

CLASSIFICATION OF FLOW

According to the magnitude of **Mach number** M = U/C where U is the velocity of body in a compressible fluid, the flows are classified as follows:

0.40 < M < 1.0	**Subsonic**
slightly less than unity < M < slightly more than unity	**Transonic**
1.0 < M < 6.0	**Supersonic**
M > 6.0	**Hypersonic**

Mach angle α is defined as

$$\alpha = \sin^{-1}\left(\frac{C}{U}\right)$$

where C is velocity of sound or pressure disturbance in compressible medium.

EQUATION OF MOTION FOR ONE DIMENSIONAL STEADY COMPRESSIBLE FLOW

Continuity Equation

$$\rho AU = \text{cons tan t} \qquad (1)$$

or

$$\frac{dA}{A} + \frac{d\rho}{\rho} + \frac{dU}{U} = 0 \qquad (2)$$

Momentum Equation

$$\text{Adp} = -\text{A}\rho\text{UdU}$$

or

$$\text{dp} - -\rho\text{UdU} \qquad (3)$$

From (1) and (3)

$$\frac{dA}{A} = \frac{dU}{U}(M^2 - 1) \qquad (4)$$

ENERGY EQUATION

Isothermal flow

$$(Z_0 - Z_1) + RT_0 \ln\frac{v_1}{v_0} = \frac{U_1^2 - U_0^2}{2g}$$

Isentropic flow

$$(Z_0 - Z_1) + \frac{k}{k-1}(p_0 v_0 - p_1 v_1) = \frac{U_1^2 - U_0^2}{2g}$$

In most cases, effect of change in elevation is negligibly small; hence the term (Z_0 - Z_1) is omitted from above equations.

Example

Find the speed of sound in oxygen at a pressure of 100 kPa (abs) and 25^{0}C. Take R = 260 J/(kg.K) and k = 1.40.

Solution

Given that k = 1.40, R = 260 J/kg.K and T = 273 + 25 = 298 K

Now $C = \sqrt{kRT} = \sqrt{1.4 \text{x} 260 \text{x} 298} = 329.4 \text{m/s}$

Example

An aeroplane is to move at Mach number of 1.5 at altitudes of 1000 m and 10, 000 m. The atmospheric pressure and densities at these elevations are:

Elevation (m)	***Pressure [kPa (abs)]***	***Density (kg/m^3)***
1000	*89.89*	*1.112*
10,000	*26.42*	*0.4125*

Calculate the speed of the plane, km/h, at these altitudes. Assume k = 1.4.

Solution

At 1000 m altitude

$$C = \sqrt{kp/\rho} = \sqrt{\frac{1.4\text{x}89890}{1.112}} = 336.4 \text{m/s}$$

Mach number $M = \frac{V}{C}$

i.e V = CM 336.4 x 1.5 = 504.6 m/s

At 10,000 m altitude

$$C = \sqrt{kp/\rho} = \sqrt{(1.4 \text{x} 26420)/0.4125} = 299.5 \text{m/s}$$

$$V = CM = 299.5 \text{x} 1.5 = 449.2 \text{m/s}$$

Example

A supersonic fighter plane flies at an altitude of 3000 m. An observer on the ground hears the sonic boom 7.5 s after the passing of the plane over head. Estimate the speed of the plane in km/h and the Mach number. (Assume R = 287 J/Kg.K and the average temperature to be 11^{0}C.

Solution

As the value of k is not given, k = 1.4 for air is assumed. Sonic velocity will be given by

$$C = \sqrt{kRT} = \sqrt{1.4x287x(273+11)} = 337.8\,m/s$$

In figure, B is the location of the plane when the sonic boom is heard at 0.

$$ABO = \alpha \text{ where } \sin\alpha = 1/M$$

If a perpendicular AD is drawn at A, then AD = Ct

In triangle AOD

$$AOD = 90 - \alpha$$

$$\therefore \quad \frac{AD}{AO} = \frac{Ct}{(AO)} = \sin(90-\alpha) = \cos\alpha = (1-\sin^2\alpha)^{1/2} = \left(1-\frac{1}{M^2}\right)^{1/2}$$

$$\frac{337.8\,x\,7.5}{3000} = \left(1-\frac{1}{M^2}\right)^{1/2}$$

$$\therefore \quad M = 1.867$$

Mach number M = V/C $\Rightarrow 1.867 = \dfrac{V}{337.8}$

i.e. V = 630.7 m/s

Problem

A supersonic plane flies at an altitude of 2500 m and 6.5 s after it has passed over the head of an observer on the ground, the sonic boom is heard. Calculate the speed of the plane and its Mach number. The average temperature of the atmosphere can be assumed as 5^{0}C. take R = 287 J/kg.K.

Answer: M = 2.02 and V = 675.2 m/s

ASSIGNMENT

Q.1. (AKTU 2023, GTU 2023, HPTU 2021, UTU 2022, 3 marks): Define Stream line, Streak line and Path line.

Answer: Described in this module.

Q.2. (RGPV 2022, 2023, 7 marks): Explain the terms:

(i) Path line ·

(ii) Streak line

(iii) Stream line

(iv) Stream tube

Answer: Described in this module.

Q.3. (BPUT 2020, 2 marks): State the types of flow line.

Answer: Described in this module.

Q.4. (AKTU 2023, GTU 2023, 2 marks): Describe laminar and turbulent flow.

Answer: Described in this module.

Q.5. (PTU 2021, 2 marks): Explain rotational and irrotational flow.

Answer: Described in this module.

Q.6. (RTU 2022, 2 marks): Define uniform and Non Uniform flow.

Answer: Described in this module.

Q.7. (AKTU 2023, 2 marks): Differentiate between steady flow and unsteady flow.

Answer: Described in this module.

Q.8. (AKTU 2023, 2 marks): Define the following:

(i) Uniform and Non Uniform Flow.

(ii) 1D, 2D, 3D flows

(iii) Compressible vs Incompressible flows

Answer: Described in this module.

Q.9. (RGPV 2022, 2023, 14 marks): Distinguish between:

(i) Steady flow and un-steady flow

(ii) Uniform and non-uniform flow

(iii) Compressible and incompressible flow

(iv) Rotational and irrigational flow

(v) Laminar and turbulent flow

Answer: Described in this module.

Q.10. (UTU 2022, 10 marks): Define following with the help of suitable mathematical expressions: (i) uniform flow, (ii) unsteady flow, (iii) Newton's Law of Viscosity, (iv) Stream Function (v) Incompressible flow.

Q.11. (JNTUH 2021, 7 marks): What is meant by one dimensional, two dimensional and three dimensional flows? Give the examples.

Answer: Described in this module.

Q.12. (GTU 2022, JNTUH 2021, 7 marks): Differentiate between: Steady and unsteady flow, Uniform and non-uniform flow, Laminar and turbulent flow, Rotational and Irrotational flow.

Answer: Described in this module.

Q.13. (BPUT 2020, 6 marks): Distinguish between (i) steady flow and un-steady flow (ii) Uniform and non-uniform flow.

Answer: Described in this module.

Q.14. (AKTU 2022, JNTUH 2021, 8 marks): Define stream function and velocity potential function.

Answer: Described in this module.

Q.15. (BPUT 2020, 2 marks): How do you relate stream function and velocity potential function?

Q.16. (AKTU 2023, 10 marks): Prove that the stream function and potential function lead to orthogonality of stream lines and equipotential flow lines.

Answer: Solved in this module.

Q.17. (AKTU 2022, 2023, 10 marks): Illustrate velocity potential and stream function. Show that 3 D continuity equation for 3 D flow in Cartesian coordinates is given by

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0$$

Answer: Described in this module.

Q.18. (AKTU 2022, GTU 2022, JNTUH 2022, 2023, 10 marks): Illustrate the derivation for continuity equation for three-dimensional flow.

Answer: Described in this module.

Q.19. (AU 2022, 2 marks): Write the uses of the Continuity Equation.

Answer: The continuity equations can be used to demonstrate the conservation of a wide range of physical phenomena, including energy, mass, momentum, natural numbers, and electric charge. The continuity equation offers useful knowledge about the flow of fluids and their behaviour as they go through a pipe or hose.

Q.20. (AU 2023, 2 marks): What are the assumption made in continuity equation?

Answer: Following are the assumptions of continuity equation:

- The tube is having a single entry and single exit
- The fluid flowing in the tube is non-viscous
- The flow is incompressible
- The fluid flow is steady

Q.21. (RGPV 2022, 2023, 7 marks): What is the physical significance of mathematical terms, Δ.v and Δxv in fluid mechanics? Where, v is the 3D velocity vector of the fluid flow.

Q.22. (BPUT 2020, 6 marks): The velocity field in a fluid medium is given by V = axi + ayj + (-2az)k. Find the equation of streamline at point P (2, 2, 4).

Answer: Solved in this module.

Q.23. (GTU 2022, 3 marks): Obtain the equation to the streamlines for the velocity field given as: $V = 2x^3i - 6x^2yj$.

Answer: Solved in this module.

Q.24. (HPTU 2022, 5 marks): The velocity components in a steady flow are u = 2kx, v = 2ky, w = -4kz. What the equation is of a stream line passing through the point (1, 0, 1)?

Answer: y = 0 and $z = 1/x^2$

Q.25. (AKTU 2023, 10 marks): Describe the method of determination of the stream function given the velocity relationship and also determine the stream function given u = 4xy and $v = c - 2y^2$.

Answer: Solved in this module.

Q.26. (AKTU 2022, 10 marks): Calculate the stream function for the given data:

(i) Velocity components; u = x – 4y and v = - y – 4x

(ii) velocity potential function $\phi = 4x(3y - 4)$.

Q.27. (JNTUH 2021, 15 marks): A uniform flow of 10 m/s is flowing over a doublet of strength 15 m^2/s. The doublet is in the line of the uniform flow. The polar co-ordinates of a point P in the flow field are 0.9 m and 30^0. Find: (a) Stream line function and (b) the resultant velocity at the point.

Answer: Solved in this module.

Q.28. (BPUT 2020, 16 marks): In a two dimensional incompressible flow, the fluid velocity components are given by u = x - 4y, v = -y - 4x. Show that the velocity potential exists and determine its form for stream function as well.

Answer: Solved in this module.

Q.29. (UTU 2022, 5 marks): Verify whether the following functions are valid potential functions:

(i) $\phi = A(x^2 - y^2)$ (ii) $\phi = A\cos x$

Q.30. (AKTU 2022, 10 marks): The velocity potential function is given by an expression

$$\phi = -\frac{xy^3}{3} - x^2 + \frac{yx^3}{3} + y^2$$

(i) Find the velocity component in x and y direction.

(ii) Show that ϕ represent a possible case of flow.

iii) Find Stream function.

Q.31. (AKTU 2022, 10 marks): For a two-dimensional flow the velocity potential function is given by the expression, $\phi = x^2 - y^2$.

(i) Determine velocity components in x and y directions.

(ii) Determine stream function.

Q.32. (AKTU 2023, RGPV 2022, 2023, 10 marks): Illustrate the properties of velocity potential function and stream function. A stream function is given by $(x^2 - y^2)$. Determine the velocity potential function of the flow.

Answer: $\psi = x^2 - y^2$; $\partial\psi/\partial x = -\partial\phi/\partial y$; $\partial\phi/\partial y = -2x$; $\therefore\ \phi = -2xy + C$

Q.33. (BPUT 2020, 6 marks): The velocity potential function is given by $5(x^2 - y^2)$. Calculate the velocity components at the point (5, 6).

Answer: Solved in this module.

Q.34. (BPUT 2020, 6 marks): The diameters of a pipe at the sections 1 and 2 are 10 cm and 15 cm respectively. Find the discharge through the pipe if the velocity of water flowing through the pipe at section 1 is 5 m/s. Also determine the velocity at section 2.

Answer: 2.4 m^3/min; 2.2 m/s

Q.35. (UTU 2022, 5 marks): The water is flowing through a tapering pipe having diameters 300 mm and 150 mm at sections 1 and 2 respectively. The discharge through the pipe is 40 litres/sec. The section 1 is 10 m above datum and section 2 is 6 m above datum. Find the intensity of pressure at section 2 if that at section 1 is 400 kN/m^2. Take density of water as 1000 kg/m^3.

Q.36. (AKTU 2023, 10 marks): In a two dimensional flow, determine a possible x component given $v = 2y^2 + 2x - 2y$. Assume steady incompressible flow.

Hint: A similar problem is solved in this module.

Q.37. (BPUT 2020, 16 marks): A fluid flow is given by $V = x^2yi + y^2zj - (2xyz + yz^2)k$, prove that it is a case of possible steady incompressible fluid flow. Calculate the velocity and acceleration at the point (2, 1, 3).

Answer: 21.587 units; 28i - 3j + 105k

Q.38. (BPUT 2020, 6 marks): A two-dimensional velocity field is given by $u = 2xy$, $v = -x^2y$. Compute (a) velocity, (b) local acceleration, (c) convective acceleration at (1,1).

Answer: A similar problem is solved in this module.

Q.39. (GTU 2023, 7 marks): Check whether the flow of liquid given by $u = 5x$ and $v = -5y$ is (i) Continuous (ii) Rotational.

Q.40. (HPTU 2021, 6 marks): If the expression for the stream function is described by $\psi = x^3 - 3xy^2$, indicate whether the flow is rotational or irrotational. If the flow is irrotational, determine the value of velocity potential. (a) $\phi = y^3 - 3xy^2$ (b) $\phi = -3x^2y$

Answer: Solved in this module.

Q.41. (HPTU 2022, 10 marks): The velocity components in a two-dimensional flow are:

$$u = 8x^2y - \frac{8}{3}y^3 \text{ and } v = -8xy^3 + \frac{8}{3}x^3$$

Show that velocity components represent a possible case of irrotational flow.

Answer: A similar problem is given in this module.

Q.42. (JNTUH 2022, 7 marks): The velocity potential function is given by an expression $\phi = y^2 - x^2 + (x^3y/3) - (xy^3/3)$. Check continuity flow. Find the velocity components in X and Y directions.

Q.43. (JNTUH 2023, 6 marks): The flow field of a fluid is given by $V = xy\,i + 2yz\,j - (yz + z^2)\,k$. Show that it represents a possible three dimensional steady incompressible continuous flow.

Q.44. (RGPV 2023, 14 marks): For the velocity field, $V = (y^3 + 6x - 3x^2y)i + (3xy^2 - 6y - x^3)j$. Check whether the flow is

(i) continuous

(ii) rotational or irrotational

(iii) if irrotational, find potential function.

COMPRESSIBLE FLOW

Q.45. (GTU 2022, 7 marks): State the Bernoulli's theorem for compressible flow and derive Bernoulli's equation when the process is isothermal.

Q.46. (GTU 2022, 3 marks): Distinguish between subsonic and supersonic flow.

Q.47. (GTU 2022, 7 marks): Derive equation for sonic velocity of sound wave in a compressible fluid in terms of the bulk modulus of elasticity of the fluid medium.

Q.48. (RGPV 2022, 7 marks): What is the significance of Mach number? Write down the short note on Fanno lines and Rayleigh lines with neat sketch.

Q.49. (GTU 2023, 7 marks): Calculate the Mach number at a point on a jet propelled aircraft which is flying at 1100 km/hour at sea level where air temperature is 20°C. Take k = 1.4 and R = 287 J/kg.°K.

Answer: C = √(rρRT/ρ) = √rRT = √(1.4 x 287 x 293 = 343.113 m/s

∴ M = V/C = 305.55/343.113 = 0.89

Q.50. (GTU 2023, 7 marks): An aeroplane as flying at a height of 10 km where temperature is -35°C. The speed of plane is corresponding to M = 2. Find the speed of the plane. Assume k = 1.4 and R = 287 J/kg °K.

Q.51. (PTU 2021, 5 marks): A gas follows the law p = (constant)ρT and flows steadily in a horizontal pipe of constant diameter. If the flow is isothermal and ratio of pressure at the two sections under consideration is p_1/p_2 = 8/7. Find the ratio V_1/V_2.